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Sensitive dependence and entropy for quantum systems

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Abstract. An attempt is made to carry to quantum mechanics the notion of sensitive dependence to initial conditions. A few simple examples and properties are described. For the entropy of quantum evolution a quantity is proposed, in the spirit of the Brin-Katok definition, which characterizes orbit complexity rather than the state reduction nature of the quantum measurement process.

1. Introduction

I have no doubt that quantum effects may cause qualitative changes in the behaviour of classically chaotic systems, as they also do for the non-chaotic systems. However, I think that some of the discussions and statements, that dismiss chaos in quantum mechanics or even point out that quantum mechanics is flawed or of limited validity for not encompassing chaos [1], are still lacking a solid basis. Take for example the much repeated statement that 'because the energy spectrum of bounded systems is discrete, wavefunctions and density matrices are almost-periodic and their predictable repetitive behaviour precludes chaos'. Going back to the works of Bocchieri-Loinger [2], Percival [3] and Krylov [4], which are in general referred to whenever there is a supporting reference for the above statement, one finds that what is correctly claimed and proved by these authors is recurrence of the wavefunction or of the density matrix, in the sense that given any $\varepsilon > 0$ and $\psi(t)$ or $\rho(t)$ there are infinitely many τ such that

$$\|\psi(t+\tau)-\psi(t)\| < \varepsilon \qquad \forall t$$

$$\|\rho(t+\tau)-\rho(t)\| < \varepsilon \qquad \forall t.$$

Recurrence is, however, also known to exist for bounded classical systems, chaotic or non-chaotic.

Chaos in classical mechanics becomes a precise notion if one identifies it with the existence of at least one positive Lyapunov exponent. It is related to the physical notion of sensitive dependence on initial conditions. The positive Lyapunov exponent measures the local rate of separation of orbits which, at time zero, differ by a small vector along the unstable manifold. The positive Lyapunov exponent does not tell us that two orbits with initial conditions differing by a small vector along the unstable manifold will not eventually come close to each other at some later time. In fact they will, with probability one, for a bounded system. Therefore the Lyapunov exponent (and sensitive dependence) are not statements about a global property like recurrence. What the Lyapunov exponent is, in fact, is the average of a local property, i.e. the average local separation of orbits in the support of some measure.

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Of course the almost-periodicity of the wavefunctions (and density matrices) of bounded quantum systems is very suggestive of regular motion. Remember however that there is nothing mysterious about it. It is simply a consequence of the linearity of the Schrödinger equation. The nonlinearity in the classical evolution is, in the quantum Schrödinger picture, traded off for the spectral problem of the Hamiltonian operator in the infinite-dimensional Hilbert space. Already if one uses the Heisenberg picture, the nonlinearity becomes manifest in the time evolution of the operators.

It is therefore clear that, to decide whether and how chaoticity (in the sense of sensitive dependence) is modified or destroyed when one passes from classical to quantum mechanics, it is not enough to invoke almost-periodicity of the wavefunctions. Instead one should see whether (and when) quantum mechanical systems are sensitively dependent on initial conditions, as an averaged local property of separation of the dynamics, starting from two close initial wavefunctions $\psi(t_0, x)$ and $\psi(t_0, x)+\delta\psi(t_0, x)$.

By the Ehrenfest theorem one knows that sufficiently localized wavepackets have to follow the classical trajectories, at least during some time. Therefore for a hyperbolic system one may find wavepackets representing locally separating orbits. However if, at time t_0, one has

$$\|\psi+\delta\psi-\psi\|_{t_0}=\delta$$

then the unitarity of the evolution operator implies that this distance remains the same for all times. The norm distance of the evolving wavefunctions remains the same at all times no matter which deformation direction one takes. Still the wavepackets must, at least during some time, be separating like the classical orbits. This only shows that statements based purely on the behaviour of the wavefunction risk being trivial and reflecting merely the linear nature of the Schrödinger equation. Non-trivial statements have necessarily to involve expectation values of operators and to make full use of their spectral properties.

Another notion that is somewhat misleading is the description of quantum mechanics as a formally integrable Hamiltonian system on infinite-dimensional phase space. This corresponds essentially to expand the wavefunction in a basis of energy eigenstates

$$\psi(t)=\sum_n a_n(t)u_n$$

and, using the fact that $|a_n(t)|=|a_n(0)|$, consider the set $|a_n(0)|$ as an infinite set of constants of motion. However, by the same reasoning, one might, in classical mechanics, define a set of initial conditions $(q(0), p(0))$ as being the constants of motion that label the orbits and declare that all classical systems are integrable.

Integrability (in the Liouville sense) means however the existence of a maximal set of analytic constants of motion in involution. The corresponding notion for quantum systems would be the existence of a complete commuting set of operators. Such a set cannot in general be constructed and, even for classically integrable systems, sometimes the constants of motion do not survive quantization.

In what concerns the question of how to characterize the complexity of quantum behaviour, I have taken in this paper a somewhat conservative point of view. This means that it is implicitly assumed that the same notions that characterize chaos and complexity in classical mechanics might, if properly defined, do the same job in quantum mechanics. It is however quite possible that sensitive dependence and entropy

are not so useful in quantum mechanics after all, and that notions like the statistical properties of energy level separation and spectral properties are the only appropriate tools. With the mathematical results available at present it is also true that, in the Hilbert space setting of quantum mechanics, one is probably better equipped to discuss robustness of eigenvalue problems than the analogue of Lyapunov stability.

2. Sensitive dependence

Ruelle [5] has studied characteristic exponents and invariant manifolds in Hilbert space. His results apply in particular to partial differential equations with bounded evolution operators. For quantum mechanics however, the somewhat trivial nature of the unitary evolution when only the states are taken into account, forces us to consider instead the evolution of matrix elements of unbounded operators [6].

Let $\psi(t)$ be a pure state in the Hilbert space $\mathcal{H}$ and $\mathcal{A}$ a symmetric operator $\{(\mathcal{A}\phi, \psi) = (\phi, \mathcal{A}\psi)\}$. The tangent space to $\mathcal{H}$ at $\psi(t)$ being $\mathcal{H}$ itself, one considers norm-preserving infinitesimal deformations $\psi_\delta(0) = \psi(0) + \delta\psi(0)$

$$\delta(\psi, \psi) = 0 \Rightarrow (\delta\psi, \psi) = 0$$

i.e. the deformations are $\delta\psi(0) = \delta \cdot \phi(0)$ with $\phi(0)$ in the orthogonal complement of $\psi(0)$ in $\mathcal{H}$ (hence also $(\phi(t), \psi(t)) = 0$). The expectation values of the operator $\mathcal{A}$ are used to measure the separation of the orbits with nearby initial conditions. Let

$$\Delta(t) = (\psi_\delta(t), \mathcal{A}\psi_\delta(t)) - (\psi(t), \mathcal{A}\psi(t)) \simeq 2\delta \cdot \mathrm{Re}(\phi(t), \mathcal{A}\psi(t)).$$

The time derivative of $\ln\|\Delta(t)\|$ measures the local rate of separation of the orbits and the average value of this separation is obtained from a time average along the orbit. Hence, whenever the following limit exists

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \frac{\partial}{\partial t} \ln|\mathrm{Re}(\phi(t), \mathcal{A}\psi(t))|\, \mathrm{d}t = \lim_{T\to\infty} \frac{1}{T} \ln|\mathrm{Re}(\phi(T), \mathcal{A}\psi(T))| = \lambda_{\mathcal{A},\phi} \tag{2.1}$$

(with $\phi(t) = \mathrm{e}^{-\mathrm{i}Ht}\phi(0) \perp \psi(t)$), it defines a *quantum characteristic exponent at ψ along the ϕ-direction.* A positive $\lambda_{\mathcal{A},\phi}$ would require exponential growth of $|\mathrm{Re}(\phi(t), \mathcal{A}\psi(t))|$. Anticipating eventual rates of growth different from exponential in quantum mechanics we define:

Definition. There is *$\mathcal{A}$-sensitive dependence at the state ψ* if there is at least a $\phi \perp \psi$ such that for any T and $M > 0$ there is a $t > T$ implying $\|\Delta(t)/\Delta(0)\| > M$.

The above definition allows for oscillations of the ratio; it only requires that $\{\|\Delta(t)/\Delta(0)\|;\, t > T\}$ be unbounded. Also, for the moment, I will not be concerned with measure considerations in Hilbert space and consider only the behaviour on the orbit $\psi(t)$.

Theorem. If $\mathcal{A}$ is defined everywhere in $\mathcal{H}$, $\mathcal{A}$-sensitive dependence cannot take place.

Proof. By the Hellinger–Toeplitz theorem a symmetric, everywhere-defined operator is necessarily bounded. If $\mathcal{A}$ is bounded, $\exists c$ such that

$$\|A\psi\| < c\|\psi\| \qquad \forall\psi \in \mathcal{H}$$

and by Schwartz's inequality $\exists M$ such that

$$|(\phi(t), \mathcal{A}\psi(t))| < M. \qquad \square$$

Henceforth I will assume that $\mathscr{A}$ is unbounded with dense domain $\mathscr{D}(\mathscr{A})$ in $\mathscr{H}$.

Sensitive dependence is a notion distinct from unboundedness of expectation values. Let $\psi(t)$ and $\phi(t)$ be states in the continuum spectrum component of a Hamiltonian H. Let $\{\chi_n\}$ be a basis in the continuum spectral subspace and $\mathscr{A}$ an operator such that

$$(\chi_{n'}, \mathscr{A}\chi_n) = f(n)\delta_{n,n'} \qquad \text{with } f'(n) > 0 \text{ and } f(n) \text{ unbounded.}$$

From $\phi(t) = \Sigma_n a_n(t)\chi_n$ and $\psi(t) = \Sigma_n b_n(t)\chi_n$, using the RAGE-theorem [7], it follows that $\forall \varepsilon > 0$ and N $\exists T$ such that

$$\frac{1}{T}\int_0^T \sum_{n \leqslant N} |a_n(T)|^2 \, \mathrm{d}t \leqslant \varepsilon.$$

Then it is easy to show [8] that $\forall \varepsilon > 0$ and $M > 0$ there is $T_{\varepsilon,M}$ such that for $T > T_{\varepsilon,M}$

$$\frac{1}{T}\int_0^T (\phi(t), \mathscr{A}\phi(t)) \, \mathrm{d}t \geqslant M(1-\varepsilon)$$

$$\frac{1}{T}\int_0^T (\psi(t), \mathscr{A}\psi(t)) \, \mathrm{d}t \geqslant M(1-\varepsilon)$$

implying that the matrix elements $(\phi(t), \mathscr{A}\phi(t))$ and $(\psi(t), \mathscr{A}\psi(t))$ are unbounded in time.

For the matrix elements involved in sensitive dependence

$$(\phi(t), \mathscr{A}\psi(t)) = \sum_n a_n^*(t) b_n(t) f(n) \tag{2.2}$$

because $\phi \perp \psi$, $\Sigma_n a_n^*(t) b_n(t) = 0$ and any bound on the sum of a finite number of terms implies the same bound on the remaining terms. Whether the matrix element (2.2) is bounded or unbounded depends on the detailed nature of the time evolution. The argument leading to unboundedness of expectation values does not imply unboundedness of the matrix elements of sensitive dependence. This suggests that these two phenomena might not always occur simultaneously.

The point spectrum of bounded quantum systems implies recurrence of the wavefunction. For sensitive dependence one has the result:

Theorem. If the energy has a point spectrum without accumulation points nor infinite degeneracies, there is no sensitive dependence for any operator $\mathscr{A}$ when restricted to a bounded energy subspace.

Proof. By bounded energy subspace of $\mathscr{H}$ we mean a subspace where vectors have components of energy less than K only. Let

$$\phi(t) = \sum_n a_n \phi_n \, \mathrm{e}^{-\mathrm{i}E_n t} \qquad \text{and} \qquad \psi(t) = \sum_n b_n \phi_n \, \mathrm{e}^{-\mathrm{i}E_n t}$$

with $E_n < K$. Then in

$$\mathrm{Re}(\phi(t), \mathscr{A}\psi(t)) = \mathrm{Re} \sum_{n,n'} a_n^* b_n (\phi_{n'}, \mathscr{A}\phi_n) \, \mathrm{e}^{-\mathrm{i}(E_n - E_{n'})t} \tag{2.3}$$

if there are no accumulation points nor infinite degeneracies of the spectrum in the range $[0, K)$, the sum on the RHS has only finitely many terms and, being a trigonometrical polynomial, is bounded for all t. □

That a point spectrum does not, by itself, exclude sensitive dependence is seen from an example. Let $\phi(0)=\Sigma_n a_n\chi_n$, $\psi(0)=\Sigma_n b_n\chi_n$

$$(\chi_k, \mathscr{A}\chi_n)=(-1)^n n\delta_{k,n+1}+(-1)^{n-1}(n-1)\delta_{k,n-1}$$

and

$$a_n=\frac{2\sqrt{6}}{\pi n}\ (n \text{ even}) \qquad a_n=0\ (n \text{ odd})$$

$$b_n=\frac{2\sqrt{2}}{\pi n}\ (n \text{ odd}) \qquad b_n=0\ (n \text{ even}).$$

Then

$$(\phi(t), \mathscr{A}\psi(t))=\frac{8\sqrt{3}}{\pi^2}\sum_{\substack{n\\ \text{even}}}\frac{1}{n}\left\{-e^{i(E_n-E_{n-1})t}+\frac{n}{n+1}e^{-i(E_{n+1}-E_n)t}\right\}. \tag{2.4}$$

If, for example

$$E_n=\frac{1}{2(n+1)}\ (n \text{ even}) \qquad E_n=\frac{1}{2n}\ (n \text{ odd})$$

the term in curly brackets on the RHS of (2.4) becomes $-\exp\{i(E_n-E_{n-1})t\}+n/(n+1)$ (n even). For $t=\pi N!!$ all (E_n-E_{n-1}) up to order $N-1$ are odd multiples of π. Hence the sum in (2.4) reaches arbitrarily high values for sufficiently large t. Notice that the sum of the remaining terms cannot cancel this growth because for large n it has a positive real part. Similar examples may easily be constructed involving, instead of an accumulation point, a sequence of arbitrarily large energy components.

Bellissard [8] had already pointed out that a pure point spectrum is not sufficient for boundedness in time of expectation values if some of the eigenstates of H are not in the domain of the operator $\mathscr{A}$. In this case however, one even notices that all eigenstates of H as well as $\psi(0)$ and $\phi(0)$ are in the domain of $\mathscr{A}$. Subtler domain questions must however be involved because under time evolution the norm $\|\mathscr{A}\psi(t)\|$ is also reaching successively higher values, approaching in some sense the boundary of the domain of $\mathscr{A}$. Notice also that $(\phi(t), \mathscr{A}\phi(t))=(\psi(t), \mathscr{A}\psi(t))=0\ \forall t$, showing that in this case sensitive dependence and unbounded growth of expectation values are different phenomena.

Notice the difference in the behaviour of $\mathrm{Re}(\phi(t), \mathscr{A}\psi(t))$ and, for example, the quantity $\|\psi(t)-\psi(0)\|$ that is used in the proof [2] of recurrence of the wavefunction

$$\|\psi(t)-\psi(0)\|=2\sum_n |b_n|^2(1-\cos E_n t).$$

Because of the normalizability of the wavefunction, this norm difference is approximated to arbitrary accuracy by a trigonometric polynomial, which is bounded and almost-periodic. In $\mathrm{Re}(\phi(0), \mathscr{A}\psi(0))$, however, the series need not be absolutely convergent. In fact, absolute convergence of the series would be a sufficient condition for boundedness.

Theorem. If $S=\Sigma_{n,n'=1}^{\infty}\|a_n^* b_n(\phi_{n'}, \mathscr{A}\phi_n)\|$ is finite then $(\phi(t), \mathscr{A}\psi(t))$ is bounded in time.

S finite $\Rightarrow \forall\varepsilon\ \exists N$ such that $\Sigma_{n,n'>N}^{\infty}\|a_n^* b_n(\phi_{n'}, \mathscr{A}\phi_n)\|<\varepsilon$. Then $\Sigma_{n,n'=1}^{N} a_n^* b_n(\phi_{n'}, \mathscr{A}\phi_n)$ is a (bounded) trigonometric polynomial that approximates $(\phi(t), \mathscr{A}\psi(t))$. □

The point nature of the spectrum, by itself, does not preclude the occurrence of quantum sensitive dependence. Notice however that the non-trivial cases necessarily involve states with arbitrarily many energy components near an accumulation point or components of arbitrarily large energy. Numerical simulation of quantum systems tend to take into account only finitely many energy components, and non-trivial quantum behaviour risks to be excluded not because of the quantum dynamics but because of the choice of initial conditions [9]. The critical nature of the initial conditions (i.e. the state $\psi(0)$ defining the orbit where sensitive dependence is studied) is apparent in the example. Indeed, if $\psi(0)$ is an arbitrary state with a finite number of energy components, then no matter how these components are chosen, $\mathrm{Re}(\phi(t), \mathscr{A}\psi(t))$ is always a trigonometric polynomial and there is no $\mathscr{A}$-sensitive dependence. The sequence $\psi_p(t) = \sum_{n=1}^{p} b_n \phi_n \, \mathrm{e}^{-\mathrm{i}E_n t}$ with $b_n \propto 1/n$ is a sequence of vectors without $\mathscr{A}$-sensitive dependence, which approaches an $\mathscr{A}$-sensitive limit point.

The phenomenon of quantum localization is related to situations where the perturbed time evolution operator has a point spectrum. However, in the literature related to the numerical simulation of quantum systems, it is pointed out as a further symptom of the stability of quantum mechanics that, even in the case of continuum spectrum, the reversibility of time evolution is not affected by the computer round-off errors. This is in contrast to the situation in the numerical simulation of classical chaotic systems. This difference in behaviour might have a simple interpretation in terms of the nature of the quantum sensitive dependence. In the classical orbits the round-off errors essentially explore the whole of the (finite-dimensional) tangent space to the orbit. In quantum mechanics however the tangent space is infinite-dimensional and the computer errors, referring to errors in a finite number of components, only explore a finite dimensional subspace of the infinite dimensional tangent space. Furthermore it seems to be usually the case that the unstable directions are associated to vectors with an infinite number of components in the unperturbed basis. Then, any error on finitely many components is simply a deformation along a direction that is not sensitively dependent and reversibility will not be affected.

I will end up with a simple example of a quantum rotator kicked by a sequence of rank-one potentials. Rank-one kicks, as compared with multiplicative potential kicks, are easier to handle analytically, allowing in particular stronger non-perturbative results [11].

Consider the following time-dependent Hamiltonian

$$\mathrm{H}(t) = -\frac{\alpha}{2\pi}\frac{\partial^2}{\partial x^2} + \lambda \sum_n |\phi_n\rangle\langle\phi_n| \delta(t-n) \tag{2.5}$$

defined on the circle $x \in [0, 1)$, where $|\phi_n\rangle$ is a sequence of Hilbert space vectors.

If all ϕ_n's were the same this would be a time-periodic Hamiltonian. Hamiltonians of the form (2.5) seem to be good testing grounds to study quantum sensitive behaviour phenomena. Here I will only illustrate its usefulness in an extremely simple case. I will be particularly concerned with the sequence

$$|\phi_n\rangle = \gamma_n \sum_{k=0}^{n} \frac{1}{2k+1} |2k\rangle \tag{2.6}$$

where γ_n is a normalization factor and $|k\rangle = \mathrm{e}^{\mathrm{i}2\pi kx}$ an eigenvector of the unperturbed Hamiltonian

$$\mathrm{H}_0(t) = -\frac{\alpha}{2\pi}\frac{\partial^2}{\partial x^2}.$$

The operator of time evolution from 'just after the $(n-1)$th kick' to 'just after the nth kick' is

$$U_n(\lambda) = e^{-i\lambda|\phi_n\rangle\langle\phi_n|} \exp\left(i\frac{\alpha}{2\pi}\frac{\partial^2}{\partial x^2}\right) = (1+\mu|\phi_n\rangle\langle\phi_n|)\exp\left(i\frac{\alpha}{2\pi}\frac{\partial^2}{\partial x^2}\right)$$
$$\mu = e^{-i\lambda} - 1. \tag{2.7}$$

Consider the (resonant) case $\alpha = 1$. The free evolution operator

$$U_0(0) = \exp\left\{i\frac{\alpha}{2\pi}\frac{\partial^2}{\partial x^2}\right\}$$

has an infinitely degenerate point spectrum, all vectors in the Hilbert space belonging to the eigenvalue one. The eigenvalue problem for each of the $U_n(\lambda)$ operators is extremely simple. The spectrum remains pointlike, with the Hilbert space splitting into two components. The vector $|\phi_n\rangle$ belongs to the eigenvalue $e^{-i\lambda}$, whereas all vectors in the orthogonal complement to $|\phi_n\rangle$ have eigenvalue one. This simple structure allows a trivial computation of the time evolution. Let for example

$$\psi(0) = c\sum_{k=1}^{\infty}\frac{1}{k^2}|2k+1\rangle. \tag{2.8}$$

Because $\psi(0)$ is orthogonal to all $|\phi_n\rangle$'s $\psi(t) = \psi(0)$ for $t \in \mathbb{Z}$. Let $\mathcal{A} = H_0$. $\psi(0)$ is not in the domain of $\mathcal{A}$ but in a rigged Hilbert space sense the matrix elements $\langle\psi(t)\mathcal{A}\psi(t)\rangle$ are well defined, finite and constant for all $t \in \mathbb{Z}$.

For the matrix elements $\langle\phi(t)\mathcal{A}\psi(t)\rangle$ with $\phi(t)\perp\psi(t)$, if $\phi(0)$ is chosen also in the subspace orthogonal to all $|\phi_n\rangle$'s, the matrix element is fixed for all times. If however $\phi(0)$ has a non-zero overlap with one $|\phi_n\rangle$, the matrix element $\langle\phi(t)\mathcal{A}\psi(t)\rangle$ grows unboundedly in time. This example may be readily extended to the study of other more complex (non-resonant) situations.

3. Entropy

In classical dynamical systems the Kolmogorov–Sinai [12, 13] entropy is an important indicator of orbit complexity. Through Pesin's theory it is closely related to the distribution of positive Lyapunov exponents, at least when the invariant measure is absolutely continuous with respect to the Lebesgue measure [14].

Several attempts have been made to carry this notion to quantum systems [15–18]. In the definition of the Kolmogorov–Sinai entropy a finite partition of a measure space plays an essential role. This is generalized to the quantum case by considering a finite dimensional Abelian subalgebra of a von Neumann algebra or an Abelian sublattice in the lattice of projections, i.e. an Abelian model of the system is constructed, the entropy being then defined as a supremum over all the Abelian models. For the purpose of comparison with a later proposal, I will first sketch a Hilbert space version of Srinivas [15, 18] construction, which is directly inspired by the Kolmogorov definition. Then notice that the resulting quantity is not purely an indicator of the dynamical diversity in the neighbourhood of an orbit, but it involves in an essential way the reduction features of the quantum measurement process. Using a cylinder measure centred on a pure state, a new quantity is defined in the spirit of the topological Brin–Katok [19] definition.

In the Kolmogorov construction one considers a partition $\mathscr{P}$ and its refinement $\bigvee_n \mathscr{P}$, the elements of $\bigvee_n \mathscr{P}$ being the non-empty intersections

$$\{a_0 \cap T^{-1}a_1 \cap \ldots \cap T^{-n}a_n\} \qquad a_i \in \mathscr{P}.$$

If μ is the invariant measure, the entropy of the partition is

$$h(T, \mathscr{P}) = -\lim_{n\to\infty} \frac{1}{n} \sum_{b\in\bigvee_n \mathscr{P}} \mu(b) \log \mu(b). \tag{3.1}$$

The entropy of the dynamics T is the supremum of $h(T, \mathscr{P})$ over all partitions. If at time zero a particular point in phase space belongs to the element $b = a_0 \cap T^{-1}a_1 \cap \ldots \cap T^{-n}a_n$ of the partition $\bigvee_n \mathscr{P}$, it means that a positive answer is assigned to the proposition '*the orbit at time k is in the element* a_k *of* $\mathscr{P}$'.

In Hilbert space quantum mechanics the set of propositions is the orthocomplemented lattice $\mathscr{L}$ of projection operators. The interpretation of the elements of $\bigvee_n \mathscr{P}$ as propositions suggests that partitions in the space of quantum events be identified with Boolean sublattices of $\mathscr{L}$. The physical meaning of the partition is provided by the spectral theorem, which associates with each self-adjoint operator $\mathscr{A}$ (observable) a spectral measure, i.e. a function from the Borel sets Δ_i on $\mathbb{R}$ to the lattice of projections $\Delta_i \to E(\Delta_i) \doteq E_i \in \mathscr{L}$. $E(\Delta_i)$ represents the measurement that selects systems with values of the observable $\mathscr{A}$ in the range Δ_i.

Because the sublattice so defined is Abelian, the set $\{E_i\}$ is a partition of unity and $E(\bigcup_i \Delta_i) = \Sigma_i E_i$ for disjoint Δ_i's, each state yields a measure on the partition, $E_i \to \mu(E_i)$, which by Gleason's theorem is defined by a density operator ρ_μ

$$\mu(E_i) = \mathrm{Tr}(\rho_\mu E_i).$$

If U_t is the unitary operator of time evolution, the projections evolve as

$$E_i^{(t)} = U_t E_i U_t^{-1}.$$

The elements of the refinement are

$$\varepsilon_{\{i_n \ldots i_0\}} = E^{(n)}(\Delta_{i_n}) \ldots E^{(1)}(\Delta_{i_1}) E(\Delta_{i_0}) \tag{3.2}$$

$\varepsilon_{\{i_n \ldots i_0\}}$ corresponds to the experiment $E(\Delta_{i_0})$ at time zero followed by $E^{(1)}(\Delta_{i_1})$ at time one, etc. However $\varepsilon_{\{i_n \ldots i_0\}}$ is a projection if and only if the projections $E^{(n)}(\Delta_{i_n})$ commute.

The measure in the refinement is

$$\mu(\varepsilon_{\{i_n \ldots i_0\}}) = \mathrm{Tr}(\rho_\mu \varepsilon^*_{\{i_n \ldots i_0\}} \varepsilon_{\{i_n \ldots i_0\}}) \tag{3.3}$$

and the following limit, whenever it exists, would be a quantum generalization of Kolmogorov's partition entropy:

$$h(\mu, \mathscr{A}) = -\lim_{n\to\infty} \frac{1}{n} \sum_{\varepsilon_{\{i_n \ldots i_0\}}} \mu(\varepsilon_{\{i_n \ldots i_0\}}) \log \mu(\varepsilon_{\{i_n \ldots i_0\}}). \tag{3.4}$$

For the entropy $h(\mu)$ one would take the supremum over the partitions induced by all self-adjoint operators in $\mathscr{H}$.

The limit in (3.4) has been proven to exist [18] if one assumes that the state μ is invariant WRT time evolution and the partition $\{E_i\}$; $\rho_\mu = U_t \rho_\mu U_t^{-1}$ and $\mu(A) = \mu(\Sigma_i E_i A E_i)$. This last condition may however be too restrictive.

At this point one notices however that in general the partition refinement as defined in (3.2) plays a role very different from $\bigvee_n \mathscr{P}$ in the classical mechanics case. In the classical case the assignment of the trajectory at a certain time, to an element of the

phase-space partition, does not change the orbit. Therefore the rate of refinement of the partition does not depend on the measurement process, but only on the diversity of orbit behaviour. In contrast, the quantum mechanical projections in (3.2) do change the dynamical state and in the end the entropy will depend as much on the non-commutativity of $\mathscr{A}$ with the time evolution U_t as on the state that defines the measure μ.

To define a quantity with a meaning closer to the classical one, we now use a construction in the spirit of the Brin-Katok [19] definition. Let ψ be a pure state and suppose we want to characterize the diversity of dynamical behaviour in a neighbourhood of ψ. Let $\mathscr{A}$ be a self-adjoint operator that as in section 2 one uses to characterize orbit separation. For the measure one chooses a Gaussian cylindrical measure centred on ψ, with covariance defined by a strictly positive operator $\mathscr{B}$ commuting with time evolution.

Let $e_i(t) = U_t e_i(0)$ be an orthonormal basis in an n-dimensional subspace contained in the orthogonal complement of $\psi(t)$ in $\mathscr{H}$. The Gaussian measure is characterized by the following finite-dimensional (cylinder) densities.

$$d\mu(\xi_1 \ldots \xi_n) = \frac{1}{(2\pi)^{n/2}} \exp\left\{ -\tfrac{1}{2} \sum_{ij} x_i^* (\Lambda^{-1})_{ij} x_j \right\} dx_1 \ldots dx_n \tag{3.5}$$

with $\xi_i = \psi + x_i e_i$ and $(\Lambda)_{ij} = (e_i, \mathscr{B} e_j)$. The μ so defined is finitely additive, but σ-additive if and only if $\mathscr{B}$ is trace-class.

As in section 2 $\mathscr{A}$-separation between nearby normalized states ψ and $\psi + \delta\phi$ is $|2\ \mathrm{Re}(\delta\phi, \mathscr{A}\psi)|$ with $\delta\phi \perp \psi$. For each finite-dimensional subspace V_n in the orthogonal complement of ψ in $\mathscr{H}$, we define the (ε, T) n-ball around ψ as the set

$$B_\varepsilon^{(n)}(T, \psi) = \{\delta\phi : |2\ \mathrm{Re}(\delta\phi(t), \mathscr{A}\psi(t))| \leqslant \varepsilon,\ 0 \leqslant t \leqslant T,\ \delta\phi \in V_n \subset \mathscr{H},\ \delta\phi \perp \psi\} \tag{3.6}$$

with $\delta\phi(t) = U_t \delta\phi$ and $\psi(t) = U_t \psi$.

The measure associated to the (ε, T) n-ball is

$$\mu(B_\varepsilon^{(n)}(T, \psi)) = (2\pi)^{-n/2} \int_{\substack{|2\mathrm{Re}(\delta\phi(t), \mathscr{A}\psi(t))| \leqslant \varepsilon \\ 0 \leqslant t \leqslant T}} dx_1 \ldots dx_n \exp\left(-\tfrac{1}{2} \sum_{ij} x_i (\Lambda^{-1})_{ij} x_j \right) \tag{3.7}$$

Provided the limits exist, the quantities

$$h^{(n)}(\psi, \varepsilon) = -\lim_{T \to \infty} \frac{1}{T} \log \mu(B_\varepsilon^{(n)}(T, \psi)) \tag{3.8}$$

$$h^{(n)}(\psi) = \lim_{\varepsilon \to 0} h^{(n)}(\psi, \varepsilon) \tag{3.9}$$

are cylinder set quantum generalizations of the Brin–Katok entropy. They characterize the complexity of the orbit structures in a neighbourhood of ψ, with the time-zero deformations restricted to a finite-dimensional subspace. Considering a sequence $V_1 \subset V_2 \subset V_3 \ldots$ of successively larger finite-dimensional subspaces, the construction of a global notion of entropy will depend on the existence of the corresponding inductive limit. No general statement can be made about the existence of these limits without specifying the nature of $\mathscr{A}$ and U_t.

The quantities $h^{(n)}(\psi, \varepsilon)$ and $h^{(n)}(\psi)$ are sensitive to the diversity of dynamical evolution in the neighbourhood of ψ, and not to the state reduction nature of the measurement process, as $h(\mu, \mathscr{A})$ in (3.4). That these quantities are indeed closer to

the corresponding classical notions is seen by the following simple estimate. The integration limits in (3.7) are defined by

$$x_i \leqslant \frac{\varepsilon}{2} \frac{1}{\sup_{0 \leqslant t \leqslant T} |\mathrm{Re}(e_i(t), \mathscr{A}\psi(t))|}.$$

For a strictly positive covariance operator $\mathscr{B}$, the Gaussian measure becomes flat in the $\varepsilon \to 0$ limit and

$$\mu(B_\varepsilon^{(n)}(T, \psi)) \simeq \prod_{i=1}^{n} \frac{\varepsilon}{\sup_{0 \leqslant t \leqslant T} |\mathrm{Re}(e_i(t), \mathscr{A}\psi(t))|}.$$

Then

$$h^{(n)}(\psi, \varepsilon) = h^{(n)}(\psi) = \lim_{T \to \infty} \frac{1}{T} \sum_{i=1}^{n} \log \sup_{0 \leqslant t \leqslant T} |\mathrm{Re}(e_i(t), \mathscr{A}\psi(t))|.$$

The only terms contributing to the limit are those where $|\mathrm{Re}(e_i(t), \mathscr{A}\psi(t))|$ is exponentially unbounded in t. Comparing with (2.1) we conclude that $h^{(n)}(\psi, \varepsilon)$ becomes the sum of the positive exponents associated to directions contained in V_n. This would be a quantum analogue of the Pesin formula. Notice however that it refers only to finite-dimensional restrictions of the entropy and the exponents.

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